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# Four-dimensional conformal field theory models with rational correlation functions

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**Abstract**
Recently established rationality of correlation functions in a globally conformal invariant quantum field theory satisfying Wightman axioms is used to construct a family of soluble models in four-dimensional Minkowski spacetime. We consider in detail a model of a neutral scalar field $\phi$ of dimension two. It depends on a positive real parameter $c$, an analogue of the Virasoro central charge, and admits for all (finite) $c$ an infinite number of conserved symmetric tensor currents. The operator product algebra of $\phi$ is shown to coincide with a simpler one, generated by a bilocal scalar field $V(x_1, x_2)$ of dimension $(1, 1)$. The modes of $V$ together with the unit operator span an infinite-dimensional Lie algebra $\mathfrak{L}_V$ whose *vacuum* (i.e. zero-energy lowest-weight) representations only depend on the central charge $c$. Wightman positivity (i.e. unitarity of the representations of $\mathfrak{L}_V$) is proven to be equivalent to $c \in \mathbb{N}$.

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## 1. Introduction

The task of constructing a conformally invariant quantum field theory model—using dressed vertices and (global) operator product expansions (OPEs)—was set forth over 30 years ago (see [9, 11, 16, 17, 19–22, 25–30, 33, 34]; for a review of this early work and further references see [36]). After a relatively quiet period (during which only some sporadic applications of the formalism appeared—see e.g. [7]) the subject was gradually revived (see [6,10,12,15,24,31,35] among others) in the wake of the two-dimensional conformal field theory (2D CFT) revolution (now the subject of textbooks—see e.g. [8] where a bibliography of original work can be found). It gathered new momentum with the discovery of the AdS–CFT correspondence and the associated intensified study of the $N = 4$ supersymmetric Yang–Mills theory (for a sample of recent papers and further references see [1, 4]).

The present work is chiefly motivated by the concept of a rational conformal field theory (RCFT). Although this notion arose in the framework of 2D CFT, recent work [23] suggests that it may be relevant to any number of spacetime dimensions. We consider in detail the simplest example beyond free fields, given in [23], the case of a model of a neutral scalar field of dimension two. More complicated (and potentially more interesting) cases involving fields of dimensions three and four are only briefly discussed.

We start by recalling the relevant results of [23] which allow us to derive the general expressions for the four-point Wightman functions.

Adding to the Wightman axioms a condition of global conformal invariance (GCI) of local observables (i.e. invariance of correlation functions under a single-valued action of the fourfold cover $G = SU(2,2)$ of the conformal group whenever $x$ and $gx$ $(g \in G)$ both belong to Minkowski space) we deduce the Huygens principle: local fields $\phi(x)$, $\psi(y)$ commute whenever the difference $x - y$ is non-isotropic; moreover,

$$[(x-y)^2]^N\,[\phi(x), \psi(y)] = 0 \qquad \text{for } N \gg 1 \tag{1.1}$$

(see [23] theorem 4.1 and proposition 4.3, where the precise bound for $N$ is given). This result is based on the fact that a spacelike separated pair of points in Minkowski space can be mapped by a proper conformal transformation into a timelike one. (Thus, GCI is a stronger requirement than invariance of Schwinger functions under the Euclidean conformal group.) The Huygens principle implies (together with energy positivity) that the Wightman distributions are rational functions of the form

$$\mathcal{W}(x_1, \ldots, x_n)\ (\equiv\langle 1, \ldots, n\rangle) = P(x_1, \ldots, x_n) \prod_{1\leqslant j<k\leqslant n} (\rho_{jk})^{-\mu_{jk}}, \tag{1.2}$$

where $P$ is a polynomial (in general, tensor valued),

$$x_{jk} \equiv x_j - x_k, \qquad \rho_{jk} = x_{jk}^2 + \mathrm{i}0\, x_{jk}^0 \quad (x^2 = \boldsymbol{x}^2 - x_0^2), \qquad \mu_{jk} \in \mathbb{Z}_+ \tag{1.3}$$

(see [23] theorem 3.1; the $\mathrm{i}0x_{jk}^0$ is only essential when $\rho_{jk}$ occurs in denominators and prescribes the contour integration for Wightman distributions that reflects energy positivity—see [32]). Hilbert space positivity is taken into account using OPE and the classification of positive-energy unitary irreducible representations of $G$ [18].

Expanding the discussion of section 5 of [23] we shall derive the general form of the truncated four-point function $\mathcal{W}_4^t(d)$ of a neutral scalar field $\phi$ of integer dimension $d$ satisfying GCI (see equation (2.1) below).

Combining proposition 5.3 and corollary 4.4 of [23] we can write

$$\begin{gathered}\mathcal{W}_4^t(d) \equiv \mathcal{W}^t(x_1, \ldots, x_4; d) = \mathcal{D}_d(\rho_{ij})\, \mathcal{P}_d(\eta_1, \eta_2) \\ \mathcal{D}_d(\rho_{ij}) = \frac{(\rho_{13}\,\rho_{24})^{d-2}}{(\rho_{12}\,\rho_{23}\,\rho_{34}\,\rho_{14})^{d-1}}, \qquad \mathcal{P}_d(\eta_1, \eta_2) = \sum_{\substack{i,j\geqslant 0\\ i+j\leqslant 2d-3}} c_{ij}\, \eta_1^i\, \eta_2^j,\end{gathered} \tag{1.4}$$

where $\eta_i$ are the conformally invariant cross ratios

$$\eta_1 = \frac{\rho_{12}\,\rho_{34}}{\rho_{13}\,\rho_{24}}, \qquad \eta_2 = \frac{\rho_{14}\,\rho_{23}}{\rho_{13}\,\rho_{24}}. \tag{1.5}$$

For $x_{jk}^2 \neq 0$ we can ignore the $\mathrm{i}0x_{jk}^0$ term in the definition of $\rho_{jk}$ (1.3). The Huygens principle (strong locality) then implies symmetry under the permutation group $\mathcal{S}_4$. Its normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ (with non-trivial elements $s_{12}\,s_{34}$, $s_{14}\,s_{23}$ and $s_{13}\,s_{24}$, where $s_{ij}$ is a substitution exchanging $i$ and $j$) acts trivially on $\eta_1$ and $\eta_2$. Hence, it suffices to impose invariance under the six-element factor group $\mathcal{S}_4/\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathcal{S}_3$ generated by

$$\begin{aligned} s_{12}:\ \mathcal{P}_d(\eta_1,\eta_2) &\mapsto \eta_2^{2d-3}\,\mathcal{P}_d\left(\frac{\eta_1}{\eta_2},\frac{1}{\eta_2}\right) = \mathcal{P}_d(\eta_1,\eta_2),\\ s_{23}:\ \mathcal{P}_d(\eta_1,\eta_2) &\mapsto \eta_1^{2d-3}\,\mathcal{P}_d\left(\frac{1}{\eta_1},\frac{\eta_2}{\eta_1}\right) = \mathcal{P}_d(\eta_1,\eta_2) \end{aligned} \tag{1.6}$$

(which also involves $s_{13} = s_{12}\,s_{23}\,s_{12} = s_{23}\,s_{12}\,s_{23}$, implying $\mathcal{P}_d(\eta_2,\eta_1) = \mathcal{P}_d(\eta_1,\eta_2)$). This leaves us with the following[4] $[\![d^2/3]\!]$ independent coefficients:

$$\begin{aligned} &c_{ij} \qquad \text{for } i \leqslant j \leqslant \frac{2d-3-i}{2}\\ &(c_{ij} = c_{ji} = c_{i,2d-3-i-j} = c_{2d-3-i-j,i} = c_{j,2d-3-i-j} = c_{2d-3-i-j,j}). \end{aligned} \tag{1.7}$$

This paper is chiefly devoted to the case $d = 2$, that is the minimal $d$ for which a non-zero truncated four-point function $\mathcal{W}_4^t(d)$ exists. We shall set in this case[5]

$$\begin{aligned} &\langle 12\rangle = \frac{c_2}{2}(12)^2, \qquad \langle 123\rangle = c_3\,(12)(23)(13),\\ &\mathcal{W}_4^t(d=2) = c_4\,(12)(23)(34)(14)\,(1+\eta_1+\eta_2),\\ &(ij) = (4\pi^2\,\rho_{ij})^{-1}. \end{aligned} \tag{1.8}$$

Parameters such as

$$c := \frac{c_2^3}{c_3^2} = 8\,\frac{\langle 12\rangle\langle 23\rangle\langle 13\rangle}{(\langle 123\rangle)^2}, \qquad c' := \frac{c_2^2}{c_4} \tag{1.9}$$

are invariant under rescaling of $\phi$. It will be proven in section 2 that if there is a single field ($\phi$) of dimension two then these constants are equal. Moreover, their common value $c$ ($=c'$) also determines the normalization of the two-point function of the stress–energy tensor and thus appears as a generalization of the Virasoro central charge. We shall then restrict our attention to the case of a single field $\phi$ corresponding to $c_2 = c_3 = c_4 = c$.

Similarly, the general truncated four-point function for $d = 3$ is

$$\begin{aligned} \mathcal{W}_4^t(3) = \frac{\rho_{13}\,\rho_{24}}{(\rho_{12}\,\rho_{23}\,\rho_{34}\,\rho_{14})^2}\{c_0(1+\eta_1^3+\eta_2^3) + c_1[(\eta_1+\eta_2)(1+\eta_1\eta_2)+\eta_1^2+\eta_2^2] + b\,\eta_1\eta_2\}\\ (c_i \equiv c_{0i} \text{ for } i = 0, 1, \text{ and } b \equiv c_{11}). \end{aligned} \tag{1.10}$$

The requirement that no $d = 2$ (scalar) field is present in the OPE of two $\phi$ in this case gives $c_1 = -c_0$ ($\neq 0$, should one demand the presence of a stress–energy tensor in the OPE).

The case $d = 4$ appears to be particularly interesting and will be briefly discussed in the concluding section 6.

The paper is organized as follows.

In section 2 we write down the OPE of two $\phi$ in terms of a bilocal scalar field $V(x_1, x_2)$ of dimension (1, 1) which satisfies—in each argument—the (free) d'Alembert equation. Using this result we sketch a proof of the statement that $V$ belongs to the OPE algebra generated by $\phi$, a property only valid in four spacetime dimensions. The free-field equations for $V$ then imply that the truncated $n$-point function of $\phi$ is expressed as a sum of one-loop diagrams with propagators $(ij)$ and a common factor $c_n$ for all $n \geqslant 4$. The uniqueness of the field $\phi$ of dimension two is proven to correspond to $c_n = c\,\alpha^n$.

[4] $[\![a]\!]$ stands for the integer part of $a$ ($[\![d^2/3]\!] = 1, 3, 5, 8$, for $d = 2, 3, 4, 5$; $[\![(d+1)^2/3]\!] - [\![d^2/3]\!] = [\![(2/3)\,(d+1)]\!] = 1, 2, 2, 3$ for $d = 1, 2, 3, 4$).

[5] The four-point Wightman function obtained from (1.8) coincides with that given by proposition 5.3 and equation (5.16) of [23] for $N_2 = c_2/32\,\pi^4$, $C_2 = c_4/(2\,\pi)^8$, $C_{20} = C_{21} = 0$.

In section 3 we establish the existence of an infinite set of conservation laws: the term with light cone singularity (12)(34) is reproduced by the contribution of an infinite number of (even-rank) conserved symmetric traceless tensor currents

$$T_{2l}(x,\zeta) = T_{\mu_1\ldots\mu_{2l}}(x)\,\zeta^{\mu_1}\cdots\zeta^{\mu_{2l}}, \qquad \Box_\zeta T_{2l}(x,\zeta) = 0 = \frac{\partial^2}{\partial x_\mu\,\partial\zeta^\mu}\,T_{2l}(x,\zeta), \tag{1.11}$$

to the OPE of two $\phi$ (including the $l = 0$ term $T_0(x) = \phi(x)$). For $\phi$ expressed as a linear combination of normal products of free fields

$$\phi(x) = \tfrac{1}{2}\sum_{i=1}^{N}\alpha_i :\varphi_i^2(x): \qquad \langle 0|\varphi_i(x_1)\,\varphi_j(x_2)|0\rangle = \delta_{ij}(12) \tag{1.12}$$

the stress–energy tensor is also given by the sum of free-field expressions:

$$T_2\,(x,\zeta) = \sum_{i=1}^{N} :\{(\zeta\cdot\partial\varphi_i(x))^2 - \tfrac{1}{2}\,\zeta^2\,\partial_\mu\varphi_i\,\partial^\mu\varphi_i + \tfrac{1}{6}\,[\zeta^2\,\Box - (\zeta\cdot\partial)^2]\varphi_i^2(x)\}: . \tag{1.13}$$

The case of equal $c_n = c$ $(n = 2, 3, \ldots)$—i.e. of a unique $\phi$—corresponds to $\alpha_i = 1$ (for $i = 1, \ldots, N$) and $c = N$. The truncated $n$-point functions of $T_2(x,\zeta)$ remain proportional to its free massless scalar field expression for all $c > 0$. Thus, the parameter $c$ indeed plays the role of a four-dimensional extension of the Virasoro central charge.

In section 4 we study the mode expansion of the bilocal field $V$, which naturally appears in the so-called analytic compact picture. We exhibit an infinite-dimensional Lie algebra $\mathfrak{L}_V$ spanned by the modes $V_{nm}(z_1, z_2)$ of $V$ and by the unit operator.

In section 5 we prove that the unitary positive-energy representations of $\mathfrak{L}_V$ correspond to positive integer $c$ (theorem 5.1). Combining this theorem with propositions 2.2 and 2.3 we derive the same result for the original field algebra of the $d = 2$ scalar field $\phi$. This implies that $\phi$ belongs to the Borchers class of a system of free fields [5] (see [32] for a text-book introduction to this concept).

Section 6 is devoted to a discussion of the results. We indicate on the way how the methods of this paper apply to fields of dimensions three and four, and end up with the formulation of two open problems.

## 2. One-loop $n$-point functions. OPE in terms of a bilocal field

We begin by rewriting the expression for the general four-point function of a neutral scalar field $\phi(x)$ of dimension two satisfying GCI in a form that suggests its generalization to the $n$-point function. According to (1.8) we have

$$\langle 1234\rangle = \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + \mathcal{W}_4^t, \qquad \left(\langle ij\rangle = \frac{c_2}{2}\,(ij)^2\right), \tag{2.1}$$

where the truncated four-point Wightman function can be written as a sum of contributions of three box diagrams:

$$\mathcal{W}_4^t = c_4\,\{(12)(34)(23)(14) + (12)(34)(13)(24) + (13)(24)(14)(23)\}. \tag{2.2}$$

This expression is reproduced by an OPE for the product of two $\phi$ that can be written compactly in terms of bilocal fields:

$$\begin{aligned} &\langle 0|\phi(x_1)\phi(x_2) = \langle 0|\{\langle 12\rangle + (12)V(x_1,x_2) + :\phi(x_1)\phi(x_2):\},\\ &V(x_1,x_2) = V(x_2,x_1), \end{aligned} \tag{2.3}$$

where the three terms are mutually orthogonal

$$\langle 0|\,V(x_1,x_2)\,|0\rangle = 0 = \langle 0|:\phi(x_1)\phi(x_2):\,|0\rangle = \langle 0|\,V(x_1,x_2):\phi(x_3)\phi(x_4):\,|0\rangle, \tag{2.4}$$

and satisfy

$$\langle 0|\, V(x_1,x_2)V(x_3,x_4)\,|0\rangle = c_4\,\{(13)(24)+(14)(23)\},$$
$$\langle 0|\, V(x_1,x_2)\phi(x_3)\,|0\rangle = c_3\,(13)(23), \tag{2.5}$$

$$\langle 0|\,:\phi(x_1)\phi(x_2):\phi(x_3)\phi(x_4)\,|0\rangle = \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + c_4\,(13)(23)(14)(24). \tag{2.6}$$

(In general, a field $V(x_1,x_2)$ is said to be bilocal if $[V(x_1,x_2), V(x_3,x_4)] = 0$ for $x_i$ spacelike to $x_j$, $i=1,2$, $j=3,4$ and if it commutes with all local fields $\phi(x_3)$ of the theory for spacelike $x_{i3}$, $i=1,2$.)

*A priori*, the algebra of $V$ and $:\phi(x_1)\phi(x_2):$ may be larger than the OPE algebra of $\phi$. This is a non-trivial result, valid only in four dimensions, that the (symmetric) bilocal fields $V(x_1,x_2)$ and $:\phi(x_1)\phi(x_2):$ can actually be determined separately from the expansion (2.3).

**Proposition 2.1.** *If $V(x_1,x_2)$ is a bilocal field obeying (2.5) then it satisfies in each argument the d'Alembert equation:*

$$\Box_1 V(x_1,x_2) = 0 = \Box_2 V(x_1,x_2), \qquad \Box_i = \frac{\partial^2}{\partial x_i^{\mu}\,\partial x_{i\,\mu}},\ i=1,2, \tag{2.7}$$

*provided the metric in the state space is positive definite.*

**Proof.** The vector-valued distribution $\Box_i V(x_1,x_2)|0\rangle$, $i=1,2$, vanishes, due to Wightman positivity since the norm squares of the corresponding smeared vectors are expressed in terms of the four-point function in the first equation (2.5). The vanishing of $\Box_i V$ then follows from local commutativity by virtue of the Reeh–Schlieder theorem. (The argument is essentially the same as the proof of the statement that the vacuum is a separating vector for local fields—see [32] section 4.) □

**Proposition 2.2.** *The bilocal field*

$$W(x_1,x_2) := 4\pi^2 x_{12}^2\,\{\phi(x_1)\phi(x_2) - \langle 12\rangle\} = V(x_1,x_2) + 4\pi^2 x_{12}^2 :\phi(x_1)\phi(x_2): \tag{2.8}$$

*allows us to determine the Taylor coefficients in $x_1$ at $x_1 = x_2$ of the two terms in the right-hand side separately.*

**Sketch of proof**[6]**.** The (pseudo)harmonicity of $V$ (2.7) implies

$$(y\,\partial_1)^n W(x_1,x)|_{x_1=x} = (y\,\partial_1)^n V(x_1,x)|_{x_1=x} + y^2\,n(n-1)4\pi^2 :[(y\,\partial_x)^{n-2}\phi(x)]\phi(x):. \tag{2.9}$$

In view of (2.7) $\Box_y (y\,\partial_1)^n V(x_1,x)|_{x_1=x} = 0$; thus $(y\,\partial_1)^n V(x_1,x)|_{x_1=x}$ appears as the harmonic part of the left-hand side of (2.9) viewed as a polynomial in $y$ and hence is uniquely determined by $W(x+y,x)$. □

It is clear from (2.5) that $V(x_1,x_2)$ is nonsingular for coinciding arguments. We can thus define a second local field

$$\phi_2(x) = \tfrac{1}{2}\,V(x,x), \tag{2.10}$$

of dimension two; it can be a multiple of $\phi(x)$ only if the ratios (1.9) coincide. Indeed, it follows from (1.8) and (2.5) that

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \langle 12\rangle, \qquad \langle 0|\phi_2(x_1)\phi(x_2)|0\rangle = \frac{c_3}{c_2}\,\langle 12\rangle,$$
$$\langle 0|\phi_2(x_1)\phi_2(x_2)|0\rangle = \frac{c_4}{c_2}\,\langle 12\rangle;$$

thus

[6] Complete proofs of propositions 2.2 and 2.3 will be published elsewhere.

$$\phi_2(x) = \lambda\,\phi(x)\left(=\frac{c_3}{c_2}\phi(x)\right) \qquad \text{implies} \qquad c_2\,c_4 = c_3^2. \tag{2.11}$$

The preceding discussion admits an extension to the $n$-point truncated function. If we set, generalizing (2.2),

$$\mathcal{W}_n^t(x_1,\ldots,x_n) = \frac{c_n}{2}\sum_{\sigma\,\in\,\mathrm{Perm}\{2\ldots n\}}(1\sigma_2)\,\overbracket{\sigma_2\sigma_3}\cdots\,\overbracket{\sigma_{n-1}\sigma_n}(1\sigma_n),$$

$$\overbracket{\sigma_i\sigma_j} = \begin{cases}(\sigma_i\sigma_j) & \text{for } \sigma_i < \sigma_j\\ (\sigma_j\sigma_i) & \text{for } \sigma_j < \sigma_i,\end{cases} \qquad n = 2,3,4,\ldots \tag{2.12}$$

then the field $\phi(x)$ of dimension two is unique if $c_n = c\,\alpha^n$ for some $\alpha > 0$, $n = 2, 3, \ldots$.

If we define $V_1$ as a linear combination of normal products of free (massless) fields,

$$V_1(x_1,x_2) = \sum_{i=1}^{N}\alpha_i : \varphi_i(x_1)\varphi_i(x_2), \tag{2.13}$$

and set $\phi(x) = \phi_1(x) = \frac{1}{2}\,V_1(x,x)$, then we can reproduce (2.12) with

$$c_n = \sum_{i=1}^{N}\alpha_i^n. \tag{2.14}$$

Furthermore, we can introduce inductively a series of bilocal and local fields $V_n(x_1, x_2)$ and $\phi_n(x)$ of dimensions (1, 1) and two setting

$$\begin{aligned} V_n(x_1,x_2) &= \lim_{x_{34}\to 0}\{4\pi^2x_{34}^2\,[V_1(x_1,x_3)V_{n-1}(x_2,x_4) - c_n\,((12)(34) + (14)(32))]\}\\ &= \sum_{i=1}^{N}\alpha_i^n : \varphi_i(x_1)\varphi_i(x_2) :, \qquad \phi_n(x) = \tfrac{1}{2}\,V_n(x,x). \end{aligned} \tag{2.15}$$

Note that the limit (2.15) is independent of the point $x_3 = x_4$ and that the field $V$ appearing in the OPE (2.3) coincides with $V_2$.

The dimension of the space of different $d = 2$ fields $\phi_k(x)$ is equal to the number of different values of $\alpha_i$ in (2.13). To see this we note that the Gram determinant of inner products

$$\langle 0|\phi_j(x_1)\phi_k(x_2)|0\rangle = \tfrac{1}{2}\,(12)^2\sum_{i=1}^{N}\alpha_i^{j+k} \tag{2.16}$$

is a multiple of $\prod_{i=1}^{N}\alpha_i^2\,\prod_{1\leqslant j<k\leqslant N}(\alpha_j-\alpha_k)^2$.

**Remark 2.1.** Fields of type (2.13) have been studied in a different context (for bounded two-dimensional fields) in [3, 13] where also infinite sums are admitted. We restrict our discussion to finite $N$ since only in this case does a stress–energy tensor exist—and is given by (1.13).

From now on we shall restrict our discussion to the simplest case of a single field $\phi$ of dimension two and set

$$c_n = c \qquad \text{for } n = 2,3,4,\ldots \tag{2.17}$$

(absorbing the possible factor $\alpha^n$ in the normalization of $\phi$).

The general form (2.12) of the truncated $n$-point function can in fact, be deduced.

**Proposition 2.3.** *Let $\phi(x)$ be a GCI Wightman field of dimension two whose truncated n-point function is given by (2.12) with $c_n = c$ for $n \leqslant 4$. Then the limit*

$$V(x_1, x_2) = \lim_{\substack{\rho_{13}\to 0\\ \rho_{23}\to 0}} (2\pi)^4 \rho_{13}\rho_{23}\{\phi(x_1)\phi(x_2)\phi(x_3) - \langle 13\rangle\phi(x_2) - \langle 23\rangle\phi(x_1) - \langle 123\rangle\} \tag{2.18}$$

*exists, does not depend on $x_3$ and defines a harmonic in each argument bilocal field $V(x_1, x_2)$. Furthermore, the truncated n-point functions of $\phi$ will be given by (2.12) for all n.*

**Sketch of proof.** Equation (2.2) and the conservation of the stress–energy tensor (see section 3) imply that (2.12) is valid for $n \leqslant 6$. The one-loop expression for the six-point function allows us to derive (2.18). The expression for the correlation function (2.5) of two $V$ satisfies the d'Alembert equation in each argument. By virtue of proposition 2.1 the operator field $V$ obeys this equation in its entire domain. Equation (2.12) for $n \leqslant 6$ implies an expansion of the form

$$\begin{aligned}\phi(x_1)\phi(x_2)\phi(x_3)\,|0\rangle &= \langle 123\rangle\,|0\rangle \\ &\quad + \sum_{\substack{i=1,2,3\\ j<k,\ j\neq i\neq k}} \{\langle jk\rangle\phi(x_i) + \overbracket{ij}\ \overbracket{ik}\ V(x_j, x_k) + (jk) : V(x_j, x_k)\phi(x_i) :\}|0\rangle \\ &\quad + :\phi(x_1)\phi(x_2)\phi(x_3):\ |0\rangle\end{aligned} \tag{2.19}$$

($(i, j, k)$ form permutations of 1, 2, 3). The result then follows. □

**Remark 2.2.** If we drop the requirement of Wightman positivity—which implies the validity of the stress–energy tensor conservation as an operator equation—then the general form of the truncated five-point function would be

$$\mathcal{W}_5^t(x_1, \ldots, x_5) = \lambda\mathcal{W}_5^t(2.12) + 4\pi^2 c\,(1-\lambda) \sum_{1\leqslant i<j\leqslant 5} \rho_{ij} \prod_{\substack{1\leqslant k\leqslant 5\\ j\neq k\neq j}} \overbracket{ik}\,\overbracket{kj}, \qquad \lambda \in \mathbb{R}. \tag{2.20}$$

We note that the one-dimensional timelike restriction $\phi(t, \mathbf{o})$ of $\phi(x)$ satisfies all properties of the chiral stress–energy tensor in a 2D CFT. It follows that all restricted truncated functions should have the form (2.12). This is satisfied by (2.20) (for our choice of constants) because of a non-trivial identity between the two terms in the one-dimensional case.

**Corollary 2.4.** *Under the assumptions of proposition 2.3 one can prove (also using proposition 2.2) that the field algebra of $\phi(x)$ coincides with the algebra of the bilocal field $V(x_1, x_2)$.*

Demanding that the truncated $n$-point function of $\phi$ for $n \geqslant 3$ is strictly less singular in $x_{ij}$ than its two-point function we have taken into account a necessary condition for Wightman positivity. We shall prove a necessary and sufficient condition for positivity in section 5.

**Remark 2.3.** If we rescale the field $\phi$ by a factor $c^{-\frac{1}{2}}$ and let $c \to \infty$ we recover the case of a generalized free field of dimension two:

$$\begin{aligned}&\text{if}\quad \hat{\phi}(x) = \frac{1}{\sqrt{c}}\phi(x) \quad \text{then} \\ &\lim_{c\to\infty} \langle 0|\,\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)\,|0\rangle = \langle 12\rangle_1\langle 34\rangle_1 + \langle 13\rangle_1\langle 24\rangle_1 + \langle 14\rangle_1\langle 23\rangle_1,\end{aligned} \tag{2.21}$$

where $\langle ij\rangle_1 = \frac{1}{2}\,(ij)^2$.

## 3. Expansion of $V(x_1, x_2)$ in local fields. Infinite set of conserved tensor currents

We shall now demonstrate that our model possesses an infinite number of conserved local tensor currents. More precisely, the bilocal field $V(x_1, x_2)$ can be expanded in a series of even-rank, conserved symmetric traceless tensor fields $T_{2l}(x, \zeta)$ (1.11) (of twist = dimension − rank = 2):

$$V(x_1, x_2) = 2 \sum_{l=0}^{\infty} C_l\, K_l(x_{12} \cdot \partial_2, \rho_{12} \Box_2) T_{2l}(x_2, x_{12}), \tag{3.1}$$

reproducing the four-point function (2.5). Here

$$K_l(s, t) = \frac{(2l + 1)!}{(l!)^2} \int_0^1 \mathrm{d}\alpha\, \alpha^l (1 - \alpha)^l \mathrm{e}^{\alpha s} \sum_{n=0}^{\infty} \frac{\left(-\frac{\alpha\,(1-\alpha)}{4}\, t\right)^n}{n!\, (2l + 1)_n}, \qquad (K_l(0, 0) = 1), \tag{3.2}$$

$\partial_2$ is the derivative in $x_2$ for fixed $x_{12}$, $\Box_2$ is the corresponding d'Alembert operator, $(\nu)_n = \Gamma(n + \nu)/\Gamma(\nu)$; it is chosen to transform the two-point function $\langle 0|T_{2l}(x_2, \zeta_2)T_{2l}(x_3, \zeta_3)|0\rangle$ into a three-point function:

$$K_l(x_{12} \cdot \partial_2,\ \rho_{12} \Box_2)\, \frac{(x_{12} \cdot r(x_{23}) \cdot \zeta)^{2l}}{\rho_{23}^{2l+2}} = \frac{(X \cdot \zeta)^{2l}}{\rho_{13}\, \rho_{23}}, \tag{3.3}$$

where

$$\begin{aligned} &\xi \cdot r(x_{23}) \cdot \zeta = \xi \cdot \zeta - 2\, \frac{(\xi \cdot x_{23})(\zeta \cdot x_{23})}{\rho_{23}}, \\ &X := X_{12}^3 := \frac{x_{13}}{\rho_{13}} - \frac{x_{23}}{\rho_{23}}, \\ &\left(X^2 = \frac{\rho_{12}}{\rho_{13}\, \rho_{23}}\right). \end{aligned} \tag{3.4}$$

In verifying (3.3) (see [10]) one applies the relation

$$\left(\frac{\Box_y}{4}\right)^n \frac{(y \cdot \zeta)^m}{(y^2)^\nu} = \frac{(\nu)_n (\nu - m - 1)_n}{(y^2)^{n+\nu}}\, (y \cdot \zeta)^m \qquad \text{for } \zeta^2 = 0$$

(used for $y = x_{23} + \alpha\, x_{12}$). In order to compute the individual contribution of $T_{2l}$ to the four-point function of $\phi$ we need the three-point function

$$\langle 0|V(x_1, x_2)T_{2l}(x_3, \zeta)|0\rangle = N_l\, C_l\, (13)(23)(X^2\zeta^2)^l C_{2l}^1(\hat{X} \cdot \hat{\zeta}), \qquad \hat{X} := \frac{X}{\sqrt{X^2}}, \tag{3.5}$$

where $N_l > 0$; $C_n^1(z)$ is the Gegenbauer polynomial satisfying

$$\left\{(1 - z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 3z\, \frac{\mathrm{d}}{\mathrm{d}z} + n(n + 2)\right\} C_n^1(z) = 0, \qquad C_n^1(1) = n + 1. \tag{3.6}$$

Writing the normalization constant in (3.5) as a product, $N_l C_l$, we exploit the fact that the three-point function vanishes whenever the structure constant $C_l = 0$.

**Remark 3.1.** The homogeneous polynomial $H_{2l}(x, \zeta) = (x^2\, \zeta^2)^l C_{2l}^1(\hat{x} \cdot \hat{\zeta})$ is the harmonic extension of the monomial $(2\, x \cdot \zeta)^{2l}$ defined on the light cone $\zeta^2 = 0$ (cf [2]):

$$\begin{aligned} &\Box_\zeta H_{2l}(x, \zeta) = (x^2)^l (\zeta^2)^{l-1} \times \left\{(1 - z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\, C_{2l}^1(z) - 3z\, \frac{\mathrm{d}}{\mathrm{d}z}\, C_{2l}^1(z) + 4l(l + 1)\, C_{2l}^1(z)\right\} = 0 \\ &\qquad (\text{for } z = \hat{x} \cdot \hat{\zeta}), \qquad H_{2l}(x, \zeta)|_{\zeta^2=0} = (2\, x \cdot \zeta)^{2l}. \end{aligned} \tag{3.7}$$

Similarly, the two-point function $\langle 0|T_{2l}(x_1, \zeta_1)T_{2l}(x_2, \zeta_2)|0\rangle$ is proportional to $\rho_{12}^{-2l-2}(\zeta_1^2\zeta_2^2)^l$ $C_{2l}^1(\hat{\zeta}_1 \cdot \hat{\zeta}_2 - 2(\hat{\zeta}_1 \cdot x_{12})(\hat{\zeta}_2 \cdot x_{12})/\rho_{12})$.

Inserting (2.3) in the four-point function (2.1) (2.2) and using (2.5) and the expansion (3.1) for $V(x_3, x_4)$ we find

$$\begin{aligned}
\langle 0|\,\phi(x_1)\phi(x_2)V(x_3,x_4)|0\rangle &= c\,(12)((13)(24)+(14)(23))\\
&= 2\sum_{l=0}^{\infty} C_l\,K_l(x_{34}\cdot\partial_4,\,\rho_{34}\,\Box_4)\langle 0|\,\phi(x_1)\phi(x_2)T_{2l}(x_4,x_{34})|0\rangle\\
&= 4\langle 12\rangle\sum_{l=0}^{\infty} N_l\,C_l^2\,\frac{(4l+1)!}{(2l)!^2}\int_0^1 \mathrm{d}\alpha\,\alpha^{2l}(1-\alpha)^{2l}\,\frac{\left(-[\alpha\,(1-\alpha)/4]\,\rho_{34}\,\Box_4\right)^n}{n!\,(2l+1)_n}\\
&\quad\times\rho_{34}^l\,(X_y^2)^{l+1}\,C_{2l}^1(\hat{X}_y\cdot\hat{x}_{34}),
\end{aligned} \tag{3.8}$$

$$X_y = \frac{x_1-y}{\rho_{1y}} - \frac{x_2-y}{\rho_{2y}}, \qquad y = x_4+\alpha\,x_{34},$$

$$\rho_{iy} = \rho_{i4}(1-\alpha)+\alpha\rho_{i3}-\alpha(1-\alpha)\rho_{34}, \qquad i=1,2.$$

It will be convenient for what follows to substitute the second conformally invariant cross ratio $\eta_2$ (1.5) by the difference $\epsilon = 1-\eta_2$, which tends to zero for $x_{34}\to 0$ (or $x_{12}\to 0$):

$$\epsilon = 1-\eta_2\;(= \mathrm{O}(x_{34}) = \mathrm{O}(x_{12})). \tag{3.9}$$

**Proposition 3.1.** *For*

$$N_l\,C_l^2 = C\begin{pmatrix}4l\\2l\end{pmatrix}^{-1} \tag{3.10}$$

*the contribution of* $V(x_3, x_4)$ *to the four-point function (2.1) is reproduced by the superposition (3.8) of three-point functions of the twist two fields* $T_{2l}$

$$\begin{aligned}
\frac{\langle 0|V(x_1,x_2)V(x_3,x_4)|0\rangle}{(13)(24)} &= c\left(1+\frac{1}{1-\epsilon}\right)\\
&= 2c\sum_{l=0}^{\infty}(4l+1)\int_0^1\left[\frac{\epsilon\,\alpha\,(1-\alpha)}{1-\epsilon\,\alpha}\right]^{2l}\frac{\mathrm{d}\alpha}{1-\epsilon\,\alpha}.
\end{aligned} \tag{3.11}$$

The *proof* of this statement is given in appendix A.

The Ward–Takahashi identity for the time-ordered three-point function of the stress–energy tensor allows us to compute the normalization $N_1 C_1$ of the Wightman function (3.5):

$$\langle 0|\phi(x_1)\phi(x_2)T_2(x_3,\zeta)\,|0\rangle = \frac{\langle 12\rangle}{3\pi^2}\,X^2(X^2\zeta^2-4(X\cdot\zeta)^2),$$
$$\text{i.e.}\quad N_1\,C_1 = -2\frac{c}{3}. \tag{3.12}$$

Comparing with (3.10) we find:

$$C_1 = -\frac{1}{4}, \qquad N_1 = 8\frac{c}{3}. \tag{3.13}$$

**Remark 3.2.** It is instructive to note that the contribution of each $T_{2l}$ to the ratio (3.11) (given by the $l$th term in the right-hand side) involves a logarithmic function in $1-\epsilon$ (see appendix A) while the infinite sum is a rational function of $\epsilon$.

## 4. The infinite-dimensional Lie algebra of field modes and its bilocal realization

The conformal compactification $\overline{M} = \mathbb{S}^3\times\mathbb{S}^1/\mathbb{Z}_2$ of Minkowski space $M = \mathbb{R}^{3,1}$ gives rise to a natural notion of conformal energy, the generator of (isometric) rotation of the timelike circle $\mathbb{S}^1$, and of an associated discrete basis of field modes. We shall parametrize

$\overline{M}$ following ([35]) in terms of complex coordinates $z = (z_a, a = 1, 2, 3, 4)$ fixed by the involution $z \mapsto z^* := \overline{z}/\overline{z}^2$:

$$\overline{M} = \left\{ z = (z_a \in \mathbb{C}, a = 1, \ldots, 4);\ z_a^* := \frac{\overline{z}_a}{\overline{z}^2} = z_a \left( z^2 = \sum_a z_a^2 =:\ \boldsymbol{z}^2 + z_4^2 \right) \right\}. \tag{4.1}$$

This condition implies the property

$$z^2\, \overline{z}^2 = 1, \qquad \frac{z_a\, z_b}{z^2} = \overline{z}_a\, z_b = z_a\, \overline{z}_b \in \mathbb{R} \qquad \text{for } z \in \overline{M} \tag{4.2}$$

which, in turn, characterizes this parametrization of $\overline{M}$. We choose the embedding map $M \subset \overline{M}$ as

$$M \ni (x^0, \boldsymbol{x}) \mapsto \boldsymbol{z} = \omega^{-1}(x)\, \boldsymbol{x}, \qquad z_4 = \frac{1 - x^2}{2\, \omega(x)}, \qquad \omega(x) = \frac{1 + x^2}{2} - \mathrm{i}\, x^0. \tag{4.3}$$

Clearly, $z$ defined by (4.3) satisfies (4.2); in particular,

$$z^2 = \frac{\overline{\omega(x)}}{\omega(x)} = \frac{(1 + \mathrm{i}\, x^0)^2 + \boldsymbol{x}^2}{(1 - \mathrm{i}\, x^0)^2 + \boldsymbol{x}^2} = \frac{1}{\overline{z}^2}, \qquad |z^2| = z \cdot \overline{z} = 1 \qquad (\text{for } z \in \overline{M}). \tag{4.4}$$

In order to write down the inverse transformation it is convenient to present $z$ in terms of a complex quaternion (or, equivalently, an element of U(2)—see [37]):

$$q\, z = z_4 + \boldsymbol{z}\, \boldsymbol{q}, \qquad q_i\, q_j = \epsilon_{ijk}\, q_k - \delta_{ij} \quad (\text{i.e. } q_1\, q_2 = -q_2\, q_1 = q_3, \text{etc}). \tag{4.5}$$

The cone at infinity, $K_\infty = \overline{M} \setminus M$, consists of the quaternions $qz \in \overline{M}$ for which $1 + qz$ is not invertible

$$q\, z \in K_\infty \quad \text{iff } 2\, \omega_z^{-1} :=\ (1 + q\, z)(1 + q^+\, z) = (1 + z_4)^2 + \boldsymbol{z}^2 = 0 (q^+\, z = z_4 - \boldsymbol{q}\, \boldsymbol{z}). \tag{4.6}$$

For $qz \notin K_\infty$ we can set

$$\begin{aligned} &\mathrm{i}\, \tilde{x} :=\ \mathrm{i}\, x_0 + \boldsymbol{q}\, \boldsymbol{x} = \frac{q\, z - 1}{q\, z + 1} \qquad \text{or} \qquad \mathrm{i}\, x_0 = \omega_z\, \frac{z^2 - 1}{2}, \\ &\boldsymbol{x} = \omega_z\, \boldsymbol{z} = \frac{2\, \boldsymbol{z}}{(1 + z_4)^2 + \boldsymbol{z}^2}. \end{aligned} \tag{4.7}$$

We shall use the fact that the flat metric on $\overline{M}$ is related to the Poincaré invariant metric on $M$ by the complex conformal factor $\omega$ (4.3)

$$\mathrm{d}z^2 = \mathrm{d}\boldsymbol{z}^2 + \mathrm{d}z_4^2 = \omega^{-2}(x)\, \mathrm{d}x^2 \qquad (\mathrm{d}x^2 = \mathrm{d}\boldsymbol{x}^2 - \mathrm{d}x_0^2). \tag{4.8}$$

To a scalar field $\phi_M(x)$ of dimension $d$ in Minkowski space we make correspond an *analytic z-picture field* $\phi(z)$ defined by

$$\phi(z) = (2\,\pi)^d\, \omega_z^d\, \phi_M(x(z)) \left( \omega_z = \frac{2}{(1 + z_4)^2 + \boldsymbol{z}^2} = \omega(x(z)) \right) \tag{4.9}$$

for $x(z)$ given by (4.7). The term *analytic* is justified by the fact that energy positivity implies analyticity of the vector-valued function $\phi(z)|0\rangle$ for $|z|^2 < 1$. Indeed, the future tube $T_+ = \{\zeta \in \mathbb{C}^4;\ \mathrm{Im}\ \zeta^0 > |\mathrm{Im}\ \underline{\zeta}|\}$, the analyticity domain of $\phi_M(\zeta)|0\rangle$ (see [32]), is mapped into a complex neighbourhood $\mathfrak{T}_+$ of the four-dimensional unit ball $\mathbb{B}^4$; more precisely, we have

$$\begin{aligned} &\mathbb{B}^4 = \{\xi \in \mathbb{R}^4;\ \xi^2 := \underline{\xi}^2 + \xi_4^2 < 1\}, \\ &\mathbb{B}^4 \times \mathbb{S}^1/\mathbb{Z}_2 = \{z = \xi \mathrm{e}^{\mathrm{i}\tau};\ \xi \in \mathbb{B}^4, \tau \in \mathbb{R}\} \subset \mathfrak{T}_+. \end{aligned} \tag{4.10}$$

Note that $\overline{M}$ appears as the boundary of the five-dimensional manifold $\mathbb{B}^4 \times \mathbb{S}^1 t / \mathbb{Z}^2$:

$$z \in \overline{M} \qquad \text{iff } z = \mathrm{e}^{\mathrm{i}\tau}\, \hat{z}, \qquad \tau \in \mathbb{R}, \qquad \hat{z} \in \mathbb{S}^3 = \{\hat{z} \in \mathbb{R}^4;\ \hat{z}^2 = 1\}. \tag{4.11}$$

The *conformal Hamiltonian* $H$ is, in this picture, nothing but the (Hermitian) generator of translation in $\tau$:

$$\mathrm{e}^{\mathrm{i}Ht}\,\phi(z)\,\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}td}\,\phi(\mathrm{e}^{\mathrm{i}t}\,z) \qquad \text{or} \qquad [H, \phi(z)] = \left(d + z_a \frac{\partial}{\partial z_a}\right)\phi(z), \qquad H|0\rangle = 0. \tag{4.12}$$

The decomposition of $\phi$ into eigenmodes of $H$ reads

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n(z), \qquad [\phi_n(z), H] = n\,\phi_n(z). \tag{4.13}$$

The *modes* $\phi_n(z)$ can be written as power series in $z_a$ and $1/z^2$ that are homogeneous in $z$ of degree $-n-d$.

For a free field $\varphi(z)$ of dimension $d = 1$ the modes $\varphi_{\pm n}$ are homogeneous harmonic polynomials spanning a space of dimension $n^2$ $\Big($as a space of $SO(4)$ symmetric traceless tensors of rank $n-1$: $\binom{n+2}{3} - \binom{n}{3} = n^2\Big)$; in particular, $\varphi_0(z) = 0$, $\varphi_1(z) = a_1/z^2$, $\varphi_{-1}(z) = a_{-1}$, $\varphi_2(z) = a_2^{\mu} z_\mu/(z^2)^2$, $\varphi_{-2}(z) = a_{-2}^{\mu} z_\mu$ etc. They are subject to the canonical commutation relations [35]

$$[\varphi_n(z), \varphi_m(w)] = \frac{(w^2)^{(n-1)/2}}{(z^2)^{(n+1)/2}}\, C^1_{|n|-1}(\hat{z}\cdot\hat{w})\,\epsilon(n)\,\delta_{n,-m}, \qquad \left(z = \sqrt{z^2}\,\hat{z}\right)$$
$$\epsilon(n) = \begin{cases} 1 & \text{for } n > 0 \\ 0 & \text{for } n = 0 \\ -1 & \text{for } n < 0. \end{cases} \tag{4.14}$$

(Here one uses the fact that the two-point function $\langle 0|\varphi(z)\varphi(w)|0\rangle = 1/(z-w)^2$ appears as a generating function for the Gegenbauer polynomials defined in (3.6).)

One can expand the bilocal field $V$ in modes $V = \sum_{\substack{n,m \\ n\neq 0\neq m}} V_{nm}$, which behave as products of $\varphi$-modes:

$$\Delta_z V_{nm}(z, w) = 0 = \Delta_w V_{nm}(z, w),$$
$$\left(z\cdot\frac{\partial}{\partial z} + n + 1\right)V_{nm}(z, w) = 0 = \left(w\cdot\frac{\partial}{\partial w} + m + 1\right)V_{nm}(z, w). \tag{4.15}$$

(The homogeneity condition only agrees with the Laplace equation if we set $V_{0m} = 0 = V_{n0}$.) The modes of the $d = 2$ field $\phi$ are most conveniently expressed as infinite sums of $V$-modes:

$$2\,\phi_n(z) = \sum_{\nu\in\mathbb{Z}} V_{\nu,n-\nu}(z, z)(V_{mn}(z, z) = V_{nm}(z, z)). \tag{4.16}$$

The components $V_{\nu,n-\nu}(z, z)$ of $\phi_n(z)$ (unlike those of $\varphi_n(z)$) span an infinite-dimensional space. This is a common feature for scalar fields of dimension $d > 1$ (more generally, for elementary conformal fields of weight $(j_1, j_2; d)$ with $d \geqslant j_1 + j_2 + 2$, in the notation of [18] and [23], which, as a result, cannot obey a free-field equation). It is all the more remarkable that the state space for a given energy $n$ is always finite-dimensional. This is a consequence of the analyticity of the vector-valued function $V_{nm}(z, w)|0\rangle$ for $z, w \in \mathfrak{T}_+$. Indeed, it then follows from (4.10) and (4.15) that

$$V_{nm}(z, w)|0\rangle = 0 \qquad \text{if } n \geqslant 0 \text{ or } m \geqslant 0. \tag{4.17}$$

Consequently, only $(n-1)$ terms of the infinite sum (4.16) contribute to the vector $\phi_{-n}(z)|0\rangle$: $2\,\phi_{-n}(z)|0\rangle = \sum_{\nu=1}^{n-1} V_{-\nu,\nu-n}(z, z)|0\rangle$.

In order to display the identity of the vacuum state spaces of $\phi$ and $V$, guaranteed by corollary 2.4, we need to include the composite twist two fields $T_{2l}(z,\zeta)$ in the operator algebra of $\phi$. Here is the realization of the four lowest-energy spaces in the two pictures. Setting for the vacuum Hilbert space

$$\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_{n=2}^{\infty} \mathcal{H}_n, \qquad (H-n)\,\mathcal{H}_n = 0 \qquad (\dim\,\mathcal{H}_0 = \dim\,\mathcal{H}_2 = 1)$$

we can write down a basis in $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ as follows:

$$\phi_{-2}\,|0\rangle = \tfrac{1}{2}\,V_{-1,-1}\,|0\rangle; \qquad \phi^a_{-3}\,z_a\,|0\rangle = V^a_{-2,-1}\,z_a\,|0\rangle\;(= z_a V^a_{-1,-2}\,|0\rangle);$$
$$\{\phi^{ab}_{-4}\,|0\rangle,\ T_2(0,\zeta)|0\rangle,\ \phi^2_{-2}\,|0\rangle\} \sim \{V^{ab}_{-2,-2}\,z_a\,z_b\,|0\rangle,\ V^{ab}_{-3,-1}\,z_a\,z_b\,|0\rangle,\ V^2_{-1,-1}\,|0\rangle\}.$$

The difficulty in describing the full state space $\mathcal{H}$ in such a manner stems from the fact that the modes of $\phi$ do not span an (infinite-dimensional) Lie algebra: the commutator $[\phi(z_1),\phi(z_2)]$ also involves all twist two conserved tensors $T_{2l}(z_2,\ z_{12})$ (and their derivatives in the first argument). $T_{2l}$ $(l=0,1,\ldots)$ together with the unit operator exhaust, in fact, the singular terms in the OPE $\phi(z)\phi(w)|0\rangle$. The resulting commutator algebra simplifies drastically for collinear $z_j = \zeta_j e$ $(e^2=1)$: it then reduces to the Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\,n(n^2-1)\delta_{n,-m} \qquad \text{for } \phi_n(\zeta\, e) = \frac{L_n}{\zeta^{n+2}},\ L_n = \phi_n(e). \tag{4.18}$$

The point is that the second argument, $z_{12}$, of $T_2$ cancels the singular factor $1/z_{12}^2$ in the OPE in the one-dimensional case.

Using the orthogonality of different quasiprimary fields we can produce a sample of projected commutation relations between $\phi_n(z)$ for non-collinear arguments illustrating the appearance of the Virasoro subalgebra as a special case.

To begin with we note that the vacuum OPE (2.3) remains valid in the $z$-picture provided we set

$$(12) = \frac{1}{z_{12}^2}\left(\text{implying } \langle 12\rangle = \frac{c}{2}\,(12)^2 = \frac{c}{2}\,(z_{12}^2)^{-2}, z^2 = \boldsymbol{z}^2 + z_4^2\right) \tag{4.19}$$

(the singularity at $z_{12}^{\,2}=0$ being treated as a limit from the domain $|z_1^2| > |z_2^2|$). Using the knowledge of the generating function for the Gegenbauer polynomials,

$$\left(\frac{1}{(z-w)^2}\right)^{\lambda} = \frac{1}{(z^2)^{\lambda}}\left(1 - 2\hat{z}\cdot\hat{w}\sqrt{\frac{w^2}{z^2}} + \frac{w^2}{z^2}\right)^{-\lambda} = \frac{1}{(z^2)^{\lambda}}\sum_{n=0}^{\infty}\left(\frac{w^2}{z^2}\right)^{n/2} C_n^{\lambda}(\hat{z}\cdot\hat{w}), \tag{4.20}$$

and the expressions (1.8) and (2.2) for two-, three- and four-point correlation functions of $\phi$, we can write the term involving the central extension of the Lie algebra generated by $\phi_n$:

$$\begin{aligned}\langle 0|\,\phi_2\,[\,\phi_n(z),\phi_{-n}(w)]\phi_{-2}\,|0\rangle &= \frac{(w^2)^{n/2-1}}{(z^2)^{n/2}}\\ &\times\langle 0|\,\phi_2\left\{C^1_{n-2}(\hat{z}\cdot\hat{w})\,\phi_0(z) + \frac{w^2}{z^2}\,C^1_n(\hat{z}\cdot\hat{w})\phi_0(w) + \frac{c}{2\,z^2}\,C^2_{n-2}(\hat{z}\cdot\hat{w})\right\}\phi_{-2}\,|0\rangle\\ &n \geqslant 1, \qquad (C^{\lambda}_{-1}\equiv 0).\end{aligned} \tag{4.21}$$

The Virasoro subalgebra (4.18) is recovered for collinear arguments noting the normalization property for Gegenbauer polynomials:

$$C_n^{\lambda}(1) = \binom{n+2\lambda-1}{n} = \frac{(2\lambda)_n}{n!}, \qquad \left(\frac{2}{\nu+1}\sum_{l=0}^{\nu} C^1_{\mu-l}(1) = 2\mu + 2 - \nu\right). \tag{4.22}$$

(In particular, equation (4.21) reproduces (4.18) for $n + m = 0$.)

The Lie algebra $\mathfrak{L}_V$ of the bilocal field $V$ is much simpler to describe. The modes $V_{nm}$ of $V$ satisfying (4.15) and the unit operator span by themselves an infinite-dimensional Lie algebra:

$$[V_{n_1n_2}(z_1, z_2), V_{n_3n_4}(z_3, z_4)] = c \prod_{j=1}^{4} (z_j^2)^{-(n_j+1)/2} \{C^1_{|n_1|-1}(\hat{z}_1 \cdot \hat{z}_3)\, C^1_{|n_2|-1}(\hat{z}_2 \cdot \hat{z}_4)\, \delta_{n_1,-n_3}\, \delta_{n_2,-n_4}$$
$$+ C^1_{|n_1|-1}(\hat{z}_1 \cdot \hat{z}_4)\, C^1_{|n_2|-1}(\hat{z}_2 \cdot \hat{z}_3)\, \delta_{n_1,-n_4}\, \delta_{n_2,-n_3}\}\epsilon(n_1)\, \epsilon(n_2)$$
$$+ (z_1^2)^{-(n_1+1)/2}(z_3^2)^{-(n_3+1)/2} C^1_{|n_1|-1}(\hat{z}_1 \cdot \hat{z}_3)\epsilon(n_1)\, \delta_{n_1,-n_3}\, V_{n_2n_4}(z_2, z_4)$$
$$+ (z_2^2)^{-(n_2+1)/2}(z_3^2)^{-(n_3+1)/2} C^1_{|n_2|-1}(\hat{z}_2 \cdot \hat{z}_3)\epsilon(n_2)\, \delta_{n_2,-n_3}\, V_{n_1n_4}(z_1, z_4)$$
$$+ (z_1^2)^{-(n_1+1)/2}(z_4^2)^{-(n_4+1)/2} C^1_{|n_1|-1}(\hat{z}_1 \cdot \hat{z}_4)\epsilon(n_1)\, \delta_{n_1,-n_4}\, V_{n_2n_3}(z_2, z_3)$$
$$+ (z_2^2)^{-(n_2+1)/2}(z_4^2)^{-(n_4+1)/2} C^1_{|n_2|-1}(\hat{z}_2 \cdot \hat{z}_4)\epsilon(n_2)\, \delta_{n_2,-n_4}\, V_{n_1n_3}(z_1, z_3). \tag{4.23}$$

This is, in fact, a central extension of the infinite-dimensional real symplectic algebra $sp(\infty, \mathbb{R})$. According to (4.16) the $\phi$-modes belong to this algebra. The vacuum representation of $\mathfrak{L}_V$ is characterized by the energy positivity condition (4.17).

The associative algebra of $V_{nm}(z,\ w)$ contains an ideal $\mathcal{I}_0$ generated by

$$\{V_{n0}(z,\ w)\, (= V_{0n}(w, z))\ ;\ n \in \mathbb{Z}\}\ (\in \mathcal{I}_0) \tag{4.24}$$

which annihilates all states in the vector space $\mathcal{H}_V$ spanned by polynomials in $V_{-n,-m}$ ($n, m \in \mathbb{N}$) acting on the vacuum. Although $\mathcal{I}_0$ may well be represented non-trivially in other sectors of the theory it is natural to work with the factor algebra $\mathcal{B}_V$ in the vacuum sector. Indeed, $\mathcal{B}_V$ can be identified as the operator algebra, generated by the bilocal field $V$, acting (non-trivially) in $\mathcal{H}_V$. The relative simplicity of the operator algebra $\mathcal{B}_V$ in $\mathcal{H}_V$ stems from the fact that the modes $V_{nm}(z, w)$ ($n \neq 0 \neq m$) are (homogeneous) harmonic functions in $z$ and $w$—see (4.15). It follows from our analysis of the mode space of the free field $\varphi(z)$ that $V_{nm}(z, w)$ span a space of dimension $n^2m^2$ except for the diagonal, $n = m$, for which the symmetry of $V$ implies that the dimension of the space is $\binom{n^2+1}{2}$.

The modes $V_{nm}$ are eigenvectors of the *Cartan elements*

$$h_l = \frac{l}{2\pi^2} \int V_{-l,l}(u, u)\, \delta(\sqrt{u^2} - 1)\, \mathrm{d}^4u, \qquad (u^2 = \boldsymbol{u}^2 + u_4^2),\ l \in \mathbb{N}. \tag{4.25}$$

(Parametrizing $u \in \mathbb{S}^3$ by $u = (\sin\psi \sin\theta \cos\varphi, \sin\psi \sin\theta, \sin\psi \cos\theta, \cos\psi)$ we can replace the volume element $\delta(\sqrt{u^2} - 1)\, \mathrm{d}^4u$ by $\sin^2\psi \sin\theta\, \mathrm{d}\psi\, \mathrm{d}\theta\, \mathrm{d}\varphi$, $0 \leqslant \psi \leqslant \pi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$; the normalization factor $1/2\pi^2$ fixes the integral (of 1) over $\mathbb{S}^3$ to 1.) We have, in particular,

$$(h_l - \delta_{lm} - \delta_{ln})V_{-n,-m}(\hat{z}, \hat{w})\, |0\rangle = 0 \qquad (\text{for } n, m \in \mathbb{N}). \tag{4.26}$$

In deriving this property one uses the relation

$$\frac{l}{2\pi^2} \int C^1_{l-1}(\hat{w} \cdot u)\, C^1_{n-1}(u \cdot \hat{z})\, \delta\big(\sqrt{u^2} - 1\big)\, \mathrm{d}^4u = \delta_{ln}\, C^1_{n-1}(\hat{w} \cdot \hat{z}). \tag{4.27}$$

It follows that the conformal Hamiltonian $H$ defined in (4.12) can be written in the form

$$H = \sum_{l=1}^{\infty} l\, h_l. \tag{4.28}$$

## 5. Unitary vacuum representations of $\mathfrak{L}_V$

We begin by introducing an anti-involution in $\mathcal{B}_V$ and the associated inner product in $\mathcal{H}$.

We define a *star operator* in the algebra of modes, setting

$$V_{nm}(z,w)^* = V_{-m,-n}(w,z) \ (= V_{-n,-m}(z,w)) \qquad \text{for } z,w \in \overline{M}, \tag{5.1}$$

so that $V(z,w)^* = V(z,w)$.

**Remark 5.1.** The anti-involution (5.1) involves a correspondence between homogeneous harmonic functions of degree $n-1$ and $-n-1$. If we write, for $n, m > 0$,

$$V_{-n,-m}(z,w) = V_{-n,-m}^{b_1\ldots b_{n-1},a_1\ldots a_{m-1}}\, z_{b_1}\ldots z_{b_{n-1}}\, w_{a_1}\ldots w_{a_{m-1}}$$

then we shall have

$$V_{-n,-m}(z,w)^* = V_{mn}(w,z) = \frac{1}{w^2\,z^2}\, V_{mn}^{a_1\ldots a_{m-1},b_1\ldots b_{n-1}}\, \frac{w_{a_1}}{w^2}\cdots\frac{w_{a_{m-1}}}{w^2}\,\frac{z_{b_1}}{z^2}\cdots\frac{z_{b_{n-1}}}{z^2},$$

where both $V_{-n,-m}$ and $V_{nm}$ are symmetric traceless tensors of rank $(n-1, m-1)$ (with respect to the indices $a_i$ and $b_j$, separately).

We shall call a Hilbert space ($\mathcal{H}$) representation of $\mathfrak{L}_V$ *unitary* if the (positive) scalar product in $\mathcal{H}$ and the conjugation (5.1) in $\mathfrak{L}_V$ are related by

$$(\,\Phi, X\,\Psi\,) = (\,X^*\,\Phi, \Psi\,) \qquad \text{for every } X \in \mathfrak{L}_V,\ \Phi,\Psi \in \mathcal{H}^F, \tag{5.2}$$

where $\mathcal{H}^F$ is the dense subspace of finite energy vectors of $\mathcal{H}$ which belongs to the domain of any $X$ in $\mathfrak{L}_V$.

One can introduce a (not necessarily positive) inner product $\langle,\rangle$ in $\mathcal{H}_V$ satisfying (5.2) defining the bra vacuum by conditions conjugate to (4.17):

$$\langle 0|V_{nm} = 0 \qquad \text{unless } n > 0 \qquad \text{and} \qquad m > 0, \tag{5.3}$$

and assuming $\langle 0|0\rangle = 1$. The main result of this section is the following characterization of the unitary vacuum representation of $\mathfrak{L}_V$.

**Theorem 5.1.** *The inner product in $\mathcal{H}_V$, defined for a (normalized) vacuum vector satisfying (4.17) and (5.3) and for $V_{nm}(z,w)$ obeying (4.23), is positive semidefinite iff $c \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$.*

**Proof.** Fix a unit vector $e \in \mathbb{S}^3$ and consider the one-dimensional subalgebra $\mathfrak{L}_V^e$ of $\mathfrak{L}_V$ generated by

$$v_{nm} := V_{nm}(e,e) \in \mathfrak{L}_V^e \subset \mathfrak{L}_V, \qquad n,m \in \mathbb{Z},\ e^2 = 1. \tag{5.4}$$

It follows from (4.23) and from (4.22) that $v_{nm}$ satisfy the commutation relations of the modes of a one-dimensional (chiral) bilocal current:

$$\begin{aligned}[][v_{n_1m_1}, v_{n_2m_2}] &= c\,n_1\,m_1(\delta_{n_1,-n_2}\,\delta_{m_1,-m_2} + \delta_{n_1,-m_2}\,\delta_{m_1,-n_2}) \\ &\quad + n_1(\delta_{n_1,-n_2}\,v_{m_1m_2} + \delta_{n_1,-m_2}\,v_{m_1n_2}) + m_1(\delta_{m_1,-n_2}\,v_{n_1m_2} + \delta_{m_1,-m_2}\,v_{n_1n_2}).\end{aligned} \tag{5.5}$$

□

**Lemma 5.2.** *There is a vector $|\Delta_n\rangle \in \mathcal{H}_V^{(n(n+1))}$ whose norm square is a multiple of $c(c-1)\cdots(c-n+1)$:*

$$\begin{aligned}\langle\Delta_n| &= \frac{1}{n!}\,\langle 0| \begin{vmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{vmatrix}, \\ \langle\Delta_n|\Delta_n\rangle &\equiv \||\Delta_n\rangle\|^2 = (n+1)!\,c(c-1)\cdots(c-n+1).\end{aligned} \tag{5.6}$$

**Proof.** It follows from (5.5) that the norm square of a polynomial of degree $n$ in $v_{kl}$ is a polynomial of degree (not exceeding) $n$ in $c$. We shall demonstrate that $\langle \Delta_n | \Delta_n \rangle$ vanishes for integer $c$ in the interval $0 \leqslant c < n$. To this end we note that if $c$ is a positive integer and $\vec{J}_m$, $m \in \mathbb{Z}$ are $c$-dimensional operator-valued vectors $\vec{J}_m = \{J^i_m,\ i = 1, \ldots, c\}$ satisfying

$$[J^i_m, J^j_n] = m\, \delta_{m,-n}\, \delta_{ij}, \qquad m, n \in \mathbb{Z},\ i, j = 1, \ldots, c, \tag{5.7}$$

then the normal products

$$v^{(c)}_{lm} =: \vec{J}_l \cdot \vec{J}_m :\equiv \sum_{i=1}^{c} : J^i_l\, J^i_m : \tag{5.8}$$

satisfy the commutation relations (5.5). If $c < n$ then $\det (v_{ij})|_{i,j=1,\ldots,n}$ appearing in the definition of $\langle \Delta_n |$, which is the Gram determinant of the scalar products of $n$ vectors in a $c$-dimensional space, should vanish. The coefficient $(n+1)!$ to the leading ($n$th) power of $c$ is computed as a sum of norm squares of terms entering the expansion of the determinant; for instance, for $n = 4$ we have

$$\begin{aligned}\lim_{c\to\infty} \left( \frac{1}{c^4} \langle \Delta_4 | \Delta_4 \rangle \right) = \frac{1}{4!^2\, c^4} \{ &\| \langle 0 | V_{11} \ldots V_{44} \|^2 + 6\, \| \langle 0 | V^2_{12}\, V_{33}\, V_{44} \|^2 \\ &+ 4\, \| \langle 0 | V_{12}\, V_{23}\, V_{13}\, V_{44} \|^2 + 3\, \| \langle 0 | V^2_{12}\, V^2_{34} \|^2 \\ &+ 3\, \| 2\, \langle 0 | V_{12}\, V_{23}\, V_{34}\, V_{14} \|^2 \} \\ = 2^4 + 6 \times 8 &+ 4 \times 8 + 3 \times 4 + 3 \times 4 = 120\, (= 5!).\end{aligned}$$

**Remark 5.2.** The Lie algebra $\mathcal{L}_V$ of bilocal modes, characterized by the commutation relations (4.23), has a reductive star subalgebra $\mathcal{U}_\infty$ (with no central extension) generated by $V_{-n,m}(z, w)$, $n, m \in \mathbb{N}$:

$$\begin{aligned}&[V_{-n_1,m_1}(\hat{z}_1, \hat{w}_1), V_{-n_2,m_2}(\hat{z}_2, \hat{w}_2)] \\ &\quad = C^1_{m_1-1}(\hat{w}_1 \cdot \hat{z}_2)\, \delta_{m_1 n_2}\, V_{-n_1,m_2}(\hat{z}_1, \hat{w}_2) \\ &\quad - C^1_{m_2-1}(\hat{w}_2 \cdot \hat{z}_1)\, \delta_{m_2 n_1}\, V_{-n_2,m_1}(\hat{z}_2, \hat{w}_1), \qquad (\hat{z}^2_i = \hat{w}^2_j = 1)\end{aligned} \tag{5.9}$$

with a central element

$$\mathcal{C}_1 = \sum_{n=1}^{\infty} h_n, \tag{5.10}$$

where $h_n$ are the Cartan operators (4.25). We have

$$\begin{aligned}[V_{-l,m}(\hat{z}, \hat{w}), \mathcal{C}_1\,] = \frac{1}{2\pi^2} \int &\{ m\, C^1_{m-1}(\hat{w} \cdot u)\, V_{-l,m}(\hat{z}, u) \\ &- l\, C^1_{l-1}(\hat{z} \cdot u)\, V_{-l,m}(u, \hat{w}) \}\, \delta(\sqrt{u^2} - 1)\, \mathrm{d}^4 u = 0,\end{aligned} \tag{5.11}$$

where we again used the relation (4.27). $\mathcal{U}_\infty$ contains what could be called the *Cartan subalgebra* of $\mathcal{L}_V$ spanned by the elements $V_{-n,n}(e, e)$ for $n \in \mathbb{N}$, $e^2 = 1$ (including $h_l$ (4.25)). $\mathcal{L}_V$ is compounded by $\mathcal{U}_\infty$, the unit element and by a pair of conjugate Abelian subalgebras $\mathfrak{L}^\pm$ (which are $\mathcal{U}_\infty$ modules), spanned by

$$\mathfrak{L}^+ \supset \{ V_{-n,-m}(z, w) \}, \qquad \mathfrak{L}^- \supset \{ V_{nm}(z, w) \}, \qquad n, m \in \mathbb{N}. \tag{5.12}$$

$\mathfrak{L}^+$ consists of positive, $\mathfrak{L}^-$ of negative, root vectors with respect to the Cartan elements $h_l$ (4.25):

$$[h_l, V_{\mp n,\mp m}(\hat{z}, \hat{w})] = \pm(\delta_{ln} + \delta_{lm})\, V_{\mp n,\mp m}(\hat{z}, \hat{w}). \tag{5.13}$$

The commutators between elements of $\mathfrak{L}^-$ and $\mathfrak{L}^+$ belong to $\mathcal{U}_\infty \cup c\mathbf{1}$. The operator

$$\mathcal{C}_2 = \sum_{n,m=1}^{\infty} \frac{n\,m}{8\pi^4} \iint V_{-n,-m}(v,u)\, V_{mn}(u,v)\; \delta(\sqrt{u^2}-1)\delta(\sqrt{v^2}-1)\, \mathrm{d}^4u\, \mathrm{d}^4v \tag{5.14}$$

commutes with $\mathcal{U}_\infty$ and should have a positive spectrum in any unitary representation of $\mathfrak{L}_V$. The counterpart of $\mathcal{C}_2$ (5.14) for the subalgebra $\mathfrak{L}_V^e$,

$$\mathcal{C}_2^e = \frac{1}{2} \sum_{n,m\geqslant 1} \frac{1}{nm}\, v_{-n,-m}\, v_{mn} \tag{5.15}$$

has its minimal eigenvalue in the subspace

$$\mathcal{H}_e^{(n)} = \{P_n(v_{-k,-l})|0\rangle;\ P_n \text{ homogeneous of degree } n \text{ in } v_{-k,-l}\}$$

on the vector $|\Delta_n\rangle$ (conjugate to) (5.6):

$$\mathcal{C}_2^e\,|\Delta_n\rangle = n(c-n+1)\,|\Delta_n\rangle, \qquad [\mathcal{C}_2^e - n(c-n+1)\,]|_{\mathcal{H}_e^{(n)}} \geqslant 0. \tag{5.16}$$

We have, for instance,

$$(\mathcal{C}_2^e - n[c+2(n-1)])\, v_{-k,\,-k}^n|0\rangle = 0 = (\mathcal{C}_2^e - n\,c)\, v_{-2n,-(2n-1)} \ldots v_{-2,-1}|0\rangle.$$

It follows from lemma 5.2 that there exist negative norm vectors unless $c$ is a positive integer. To prove that for $c \in \mathbb{N}$ the vacuum representation of $\mathfrak{L}_V$ is indeed unitary it suffices to note that in this case $V$ can be written in the form

$$V(z_1, z_2) = \sum_{i=1}^{c} :\varphi_i(z_1)\varphi_i(z_2): \tag{5.17}$$

where $\varphi_i$ are mutually commuting free zero-mass fields and to recall that a system of free fields satisfies all Wightman axioms (including positivity). □

We have established on the way the following result (as a direct consequence of lemma 5.2).

**Proposition 5.3.** *The vacuum representation of the infinite-dimensional Lie algebra $\mathfrak{L}_v$ of the two-dimensional bilocal chiral field*

$$v(z,w) = \frac{1}{zw} \sum_{n,m\in\mathbb{Z}} v_{nm}\, z^{-n}\, w^{-m}, \qquad z, w \in \mathbb{C} \tag{5.18}$$

*whose modes satisfy (5.5) (and $v_{mn}\,|0\rangle = 0$ unless $m < 0$ and $n < 0$) is only unitary for positive integer $c$.*

This is an analogue of the Kac–Radul theorem [14] on the unitary representations of the $W_{1+\infty}$ algebra. It is clear that the algebra of the two-dimensional stress tensor

$$T(z) = \tfrac{1}{2}\, v(z,z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}, \tag{5.19}$$

i.e. the Virasoro algebra (4.18), is a true subalgebra of $\mathfrak{L}_v$ since it admits unitary representations for all $c \geqslant 1$ as well as a discrete series for $c = c_n = 1 - 6/(n+1)(n+2)$ $(n = 1, 2, \ldots,$ the unitary Virasoro module $\mathcal{H}_{c_n}$ being the quotient space of the corresponding lowest-weight module with respect to a singular vector at 'level' (= eigenvalue of $L_0$) $n(n+1)$).

The situation is different for $D = 4$ since $V(z,w)$ is harmonic in each argument in this case. Due to corollary 2.4 the algebra $\mathcal{B}_V$ is then not bigger than the original OPE algebra of $\phi$, so the result of theorem 5.1 extends to it.

**Corollary 5.4.** *Under the assumptions of proposition 2.3 it follows from theorem 5.1 that the quantum theory of the field $\phi$ with truncated n-point function (2.12) satisfies Wightman positivity iff c is a natural number (in which case $\phi$ belongs to the Borchers class of a set of free fields).*

## 6. Extensions of the results. Concluding remarks

The preceding results—and methods—apply to fields of higher dimension and arbitrary tensor structure. We shall establish important special cases of the following

**Conjecture.** *If a neutral tensor field of integer dimension has truncated n-point functions which are multiples of the corresponding correlators of normal products of (derivatives of) free fields for $n \leqslant 6$, then Wightman positivity implies that the proportionality constant is a positive integer. (As indicated in section 2, for $d = 2$ the statement follows from the expression for the four-point function.)*

Our first example is a conserved current whose (first five) truncated correlation functions are obtained from those of the current of a system of two-component spinors,

$$J^{\mu}(x;\, c_{\psi}) = \sum_{j=1}^{c_{\psi}} :\psi_j^*(x)\tilde{\sigma}^{\mu}\psi_j(x):, \qquad (-\tilde{\sigma}^0) = \tilde{\sigma}_0 = \mathbb{I} = \sigma_0,\ \tilde{\sigma}^j = -\sigma^j = -\sigma_j, \tag{6.1}$$

by substituting the positive integer $c_{\psi}$ by an arbitrary real number. Here $\psi_j$ are mutually anticommuting free Weyl fields:

$$\langle 0|\, \psi_j(x_1)\, \psi_k^*(x_2)\, |0\rangle = \delta_{jk}\, S(x_{12}), \qquad S(x_{12}) = \mathrm{i}\, \underset{\sim}{\partial}_2(12) = \mathrm{i}\, \frac{\underset{\sim}{x}_{12}}{2\,\pi^2\, \rho_{12}^2}, \tag{6.2}$$

and we have used the conventions

$$\underset{\sim}{\partial}_2 = \sigma_{\mu}\, \frac{\partial}{\partial x_{2\,\mu}} (\underset{\sim}{x} = \sigma_{\mu} x^{\mu}), \qquad \sigma^{\mu}\, \tilde{\sigma}_{\nu} + \sigma_{\nu}\, \tilde{\sigma}^{\mu} = -2\, \delta^{\mu}_{\nu}. \tag{6.3}$$

Introducing the spin-tensor components of the current

$$J(x)(= J_{\alpha\,\dot{\beta}}(x)) = \tfrac{1}{2}\, \sigma_{\mu} J^{\mu} \left( = \sum_{j=1}^{c_{\psi}} :\psi_{j\alpha}(x)\, \psi_{j\dot{\beta}}^*(x): \right) \tag{6.4}$$

we can write

$$\begin{aligned} \langle 0|\, J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2)\, |0\rangle &= c_{\psi}\, S_{\alpha_1\,\dot{\beta}_2}(x_{12})\, {}^t S_{\dot{\beta}_1\,\alpha_2}(x_{12}) \\ &= c_{\psi} \left\{ S_{\alpha_1\,\dot{\beta}_2}(x_{12})\, S_{\alpha_2\,\dot{\beta}_1}(x_{12}) - \frac{\epsilon_{\alpha_1\,\alpha_2}\epsilon_{\dot{\beta}_1\,\dot{\beta}_2}}{4\,\pi^4\, \rho_{12}^3} \right\}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2) - \langle 0|\, J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2)\, |0\rangle &= {}^t S_{\dot{\beta}_1\,\alpha_2}(x_{12})\, V_{\alpha_1\,\dot{\beta}_2}(x_1, x_2) \\ &+ S_{\alpha_1\,\dot{\beta}_2}(x_{12})\, {}^t V_{\dot{\beta}_1\,\alpha_2}(x_1, x_2) + :J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2):, \end{aligned} \tag{6.6}$$

where ${}^t S$ (${}^t V$) stands for the transposition of $S$ ($V$). Multiplying both sides by $(2\,\pi^2/\mathrm{i})\, \rho_{12}\, \tilde{x}_{12}^{\dot{\beta}_2\alpha_2}$ and setting

$$W_{\alpha_1\,\dot{\beta}_1}(x_1, x_2) = \frac{2\,\pi^2}{\mathrm{i}}\, \rho_{12}\, \tilde{x}_{12}^{\dot{\beta}_2\alpha_2} \{J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2) - \langle 0|\, J_{\alpha_1\,\dot{\beta}_1}(x_1)\, J_{\alpha_2\,\dot{\beta}_2}(x_2)\, |0\rangle\} \tag{6.7}$$

we obtain

$$W_{\alpha_1\,\dot{\beta}_1}(x_1, x_2) = V_{\alpha_1\,\dot{\beta}_1}(x_1, x_2) + {}^t V_{\dot{\beta}_1\,\alpha_1}(x_1, x_2) + :J_{\alpha_1\,\dot{\beta}_1}(x_1)\, \mathrm{tr}(\tilde{x}_{12}\, J(x_2)): \tag{6.8}$$

where the bilocal field $V$ satisfies

$$\langle 0|\, V_{\alpha_1\,\dot{\beta}_1}(x_1, x_2)\, V_{\alpha_2\,\dot{\beta}_2}(x_3, x_4)\, |0\rangle = S_{\alpha_1\,\dot{\beta}_2}(x_{14})\, {}^t S_{\dot{\beta}_1\,\alpha_2}(x_{23}). \tag{6.9}$$

It follows from (6.9) that

$$\tilde{\partial}_1^{\dot{\alpha}\alpha_1} V_{\alpha_1\dot{\beta}_1}(x_1,x_2) = 0 = V_{\alpha_1\dot{\beta}_1}(x_1,x_2)\frac{\overleftarrow{\partial}}{\partial x_{2\,\mu}}\tilde{\sigma}_\mu^{\dot{\beta}_1\beta}. \tag{6.10}$$

As a result $V_{\alpha\dot{\beta}}$ and the normal product of $J$ appearing in the right-hand side of (6.9) can be determined separately and we can prove as in section 5 that $c_\psi(c_\psi - 1)\cdots(c_\psi - n + 1) \geqslant 0$ for $n = 1, 2, \ldots$.

As a second example we consider the Lagrangean density

$$\mathcal{L}_F(x) = -\tfrac{1}{4}\sum_{a=1}^{c_F} :F^a_{\mu\nu}(x)\,F_a^{\mu\nu}(x):\ (c_F \in \mathbb{N}) \tag{6.11}$$

and the associated analytic continuation of truncated Wightman functions to arbitrary positive real $c_F$. The truncated $n$-point function of $\mathcal{L}_F$ can again be written as a sum of $\frac{1}{2}(n-1)!$ one-loop graphs, the propagator associated with a line joining the vertices 1 and 2 being

$$\begin{aligned}\mathcal{D}_{\lambda_1\mu_1\lambda_2\mu_2}(x_{12}) &= \frac{1}{4}\{\partial_{\lambda_1}(\partial_{\lambda_2}\eta_{\mu_1\mu_2} - \partial_{\mu_2}\eta_{\mu_1\lambda_2}) - \partial_{\mu_1}(\partial_{\lambda_2}\eta_{\lambda_1\mu_2} - \partial_{\mu_2}\eta_{\lambda_1\lambda_2})\}\frac{1}{4\pi^2\rho_{12}}\\ &= \frac{r_{\lambda_1\lambda_2}(x_{12})\,r_{\mu_1\mu_2}(x_{12}) - r_{\lambda_1\mu_2}(x_{12})\,r_{\mu_1\lambda_2}(x_{12})}{4\pi^2\rho_{12}^2}.\end{aligned} \tag{6.12}$$

This expression for the propagator also enters the OPE of two $\mathcal{L}$ (together with a tensor-valued bilocal field):

$$\begin{aligned}\langle 0|\mathcal{L}_F(x_1)\mathcal{L}_F(x_2) &= \langle 0|\{2\,c_F\,\mathcal{D}^{\lambda_1\mu_1\lambda_2\mu_2}(x_{12})\,\mathcal{D}_{\lambda_1\mu_1\lambda_2\mu_2}(x_{12})\\ &\quad + \mathcal{D}^{\lambda_1\mu_1\lambda_2\mu_2}(x_{12})\,V_{\lambda_1\mu_1\lambda_2\mu_2}(x_1,x_2) + :\mathcal{L}_F(x_1)\mathcal{L}_F(x_2):\},\\ 2\,\mathcal{D}^{\lambda_1\mu_1\lambda_2\mu_2}\mathcal{D}_{\lambda_1\mu_1\lambda_2\mu_2} &= \frac{3}{(\pi\,\rho_{12})^4}.\end{aligned} \tag{6.13}$$

For $c \in \mathbb{N}$, $V$ has a realization as a sum of normal products of free Maxwell fields:

$$V_{\lambda_1\mu_1\lambda_2\mu_2}(x_1,x_2) =: F_{\lambda_1\mu_1}^{\ a}(x_1)F_{\lambda_2\mu_2}^{\ a}(x_2):. \tag{6.14}$$

The OPE (6.13) allows us to compute the truncated four-point function of $\mathcal{L}_F$ which appears as a special case of the five-parameter expression $\mathcal{W}_4^t(d = 4)$ computed from equations (1.4)–(1.7):

$$\begin{aligned}\mathcal{W}_4^t(4) = \frac{\rho_{13}^2\,\rho_{24}^2}{\rho_{12}^3\,\rho_{23}^3\,\rho_{34}^3\,\rho_{14}^3}&\{c_0(1+\eta_1^5+\eta_2^5) + c_1(\eta_1+\eta_2+\eta_1^4+\eta_2^4+\eta_1\,\eta_2\,(\eta_1^3+\eta_2^3))\\ &+ c_2(\eta_1^2+\eta_2^2+\eta_1^3+\eta_2^3+\eta_1^2\,\eta_2^2\,(\eta_1+\eta_2))\\ &+ b_1\,\eta_1\,\eta_2(1+\eta_1^2+\eta_2^2) + b_2\,\eta_1\,\eta_2(\eta_1\,\eta_2+\eta_1+\eta_2)\}\end{aligned}$$

$$(c_i \equiv c_{0i} \text{ for } i = 0,1,2;\ b_i \equiv c_{1i} \text{ for } i = 1,2). \tag{6.15}$$

Indeed the contribution $\mathcal{W}_\square$ of the box diagram (computed by using formulae for traces of products of $r^\mu_\nu$ given in appendix B),

$$\begin{aligned}\mathcal{W}_\square &= c_F\,\mathcal{D}^{\lambda_1\ \ \lambda_2}_{\ \ \mu_1\ \ \mu_2}(x_{12})\,\mathcal{D}_{\lambda_1\ \ \lambda_4}^{\ \ \mu_1\ \ \mu_4}(x_{14})\,\mathcal{D}_{\lambda_2\ \ \lambda_3}^{\ \ \mu_2\ \ \mu_3}(x_{23})\,\mathcal{D}^{\lambda_3\ \ \lambda_4}_{\ \ \mu_3\ \ \mu_4}(x_{34})\\ &= 32\,c_F\,\frac{1}{(2\pi)^4(\rho_{12}\rho_{23}\rho_{34}\rho_{14})^2}\left(1+\frac{\eta_1}{\eta_2}+\frac{\eta_2}{\eta_1}-\frac{2}{\eta_2}-\frac{2}{\eta_1}+\frac{1}{\eta_1\eta_2}\right),\end{aligned} \tag{6.16}$$

which enters the expression for the truncated four-point function of $\mathcal{L}(x)$

$$\begin{aligned}\mathcal{W}_4^t &= (1+s_{12}+s_{23})\,\mathcal{W}_\square(x_1,x_2,x_3,x_4)\\ &= \mathcal{W}_\square(x_1,x_2,x_3,x_4) + \mathcal{W}_\square(x_2,x_1,x_3,x_4) + \mathcal{W}_\square(x_1,x_3,x_2,x_4)\end{aligned} \tag{6.17}$$

fits the expression (6.15) for

$$c_0 = c_2 = b_1 = -\frac{1}{2}\,c_1 = \frac{c_F}{8\,\pi^8}, \qquad b_2 = 0. \tag{6.18}$$

The first local field in the expansion of $V$ around the diagonal is the stress–energy tensor:

$$T^{\mu}_{\nu} = \frac{1}{4}\, V^{\kappa\lambda}{}_{\kappa\lambda}(x,x)\delta^{\mu}_{\nu} - V^{\lambda\mu}{}_{\lambda\nu}(x,x) = -\mathcal{L}(x)\delta^{\mu}_{\nu} - V^{\lambda\mu}{}_{\lambda\nu}(x,x). \tag{6.19}$$

Conversely, the bilocal tensor field $V^{\lambda_1\mu_1}{}_{\lambda_2\mu_2}(x_1,x_2)\,(= -V^{\mu_1\lambda_1}{}_{\lambda_2\mu_2}(x_1,x_2) = -V^{\lambda_1\mu_1}{}_{\mu_2\lambda_2}(x_1,x_2))$ appears in the OPE of two $T^{\mu}_{\nu}$ and can be determined from it in two steps. First, one derives the formula

$$\begin{aligned}\langle 0|\, V^{\lambda_1\mu_1}{}_{\lambda_2\mu_2}(x_1,x_2)\, V^{\lambda_3\mu_3}{}_{\lambda_4\mu_4}(x_3,x_4)\,|0\rangle &= c_F\, \mathcal{D}^{\lambda_1\mu_1\lambda_3\mu_3}(x_{13})\mathcal{D}_{\lambda_2\mu_2\lambda_4\mu_4}(x_{24})\\ &\quad + c_F\, \mathcal{D}^{\lambda_1\mu_1}{}_{\lambda_4\mu_4}(x_{14})\mathcal{D}^{\lambda_3\mu_3}{}_{\lambda_2\mu_2}(x_{23})\end{aligned} \tag{6.20}$$

and deduces from it that $V^{\lambda_1\mu_1}{}_{\lambda_2\mu_2}(x_1,x_2)$ satisfies in each argument the free Maxwell equations. Secondly, one uses this fact to single out the contribution of $V$ in the OPE of two $T$. Once more Wightman positivity implies $c_F \in \mathbb{N}$.

**Remark 6.1.** The use of different notation, $c(= c_{\phi})c_{\psi}$ and $c_F$ for the constants multiplying the truncated functions of normal products of the free fields $\phi$, $\psi$ and $F_{\mu\nu}$, respectively, is justified by the fact that they correspond to (and exhaust the) different tensor structures in the general conformal invariant three-point function of the stress–energy tensor [31].

At the same time the four-point functions of the conserved current $J_{\mu}$ and $\mathcal{L}(x)$ involve structures which cannot be reduced to normal products of free fields. If, for instance, $b_2 \neq 0$ in (6.15) the three-point function of $\mathcal{L}(x)$ will not vanish (unlike the case of superposition of type (6.11) of normal products of free Maxwell fields). More generally, we have a four-parameter family of admissible four-point functions of $\mathcal{L}(x)$ obtained from equation (6.15) with the restriction

$$c_2 = -c_0 - c_1\ (\neq 2\, c_0) \tag{6.21}$$

coming from the requirement that no $d = 2$ field appears in the OPE of $\mathcal{L}(x_1)\mathcal{L}(x_2)$ (and that the stress–energy tensor is present in this OPE). They are only compatible with three-point functions of $T$ of the type (6.19) (i.e. with the third of the three admissible structures in this three-point function given in [31]—cf remark 6.1).

To summarize: looking for a 4D RCFT beyond the Borchers' class of free fields we have excluded the theory of a bilocal field of dimension $(1, 1)$ and have come to the following problem. Assume that the only local fields in the observable algebra, satisfying GCI, of dimension $d \leqslant 4$ are the (conserved traceless) stress–energy tensor $T_{\mu\nu}(x)$ and a scalar field $\mathcal{L}(x)$ of dimension four (playing the role of an action density). The problem is to construct an OPE algebra consistent with the $n$-point functions of these fields for $n \leqslant 4$ that would allow us to compute higher point correlation functions and to implement the condition of Wightman positivity. This example is attractive because the dimensions of the basic fields $\mathcal{L}$ and $T_{\mu\nu}$ are protected. Moreover, in any renormalizable quantum field theory one can define a (gauge-invariant) local action density and a stress–energy tensor.

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## Appendix A. Proof of proposition 3.1

We shall first compute the sum in the right-hand side of (3.8) for

$$
\begin{aligned}
&\rho_{34} = 0,\\
&2\, X_y \cdot x_{34} = \frac{\rho_{14} - \rho_{13}}{(1-\alpha)\rho_{14} + \alpha\,\rho_{13}} - \frac{\rho_{24} - \rho_{23}}{(1-\alpha)\rho_{24} + \alpha\,\rho_{23}}\\
&\qquad = \frac{\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24}}{[(1-\alpha)\rho_{14} + \alpha\,\rho_{13}][(1-\alpha)\rho_{24} + \alpha\,\rho_{23}]}\\
&\qquad = \frac{-\epsilon}{\left[\alpha + (1-\alpha)\rho_{14}/\rho_{13}\right]\left[1 - \alpha + \alpha\,\rho_{23}/\rho_{24}\right]},\\
&\frac{\rho_{13}\,\rho_{24}}{\rho_{12}}\, X_y^2 = \frac{1}{\left[\alpha + (1-\alpha)\rho_{14}/\rho_{13}\right]\left[1 - \alpha + \alpha\,\rho_{23}/\rho_{24}\right]},
\end{aligned}
\tag{A.1}
$$

and then use the result to give a general proof of proposition 3.1. According to (A.1) we have

$$
\lim_{\rho_{34}\to 0} \left(\rho_{34} X_y^2\right)^l C_{2l}^1(\hat{X}_y \cdot \hat{x}_{34}) = \frac{\epsilon^{2l}}{\left[\alpha + (1-\alpha)\rho_{14}/\rho_{13}\right]^{2l}\left[1 - \alpha + \alpha\,\rho_{23}/\rho_{24}\right]^{2l}}.
$$

Conformal invariance allows us to send $x_1$ to infinity, setting $\rho_{14}/\rho_{13} \to 1$, $\rho_{23}/\rho_{24} \to 1 - \epsilon$, thus reproducing the right-hand side of (3.11). Taking the sum in $l$ we reduce the proof of (3.11) to verifying the identity

$$
2\int_0^1 \frac{(1-\epsilon\,\alpha)[(1-\epsilon\,\alpha)^2 + \epsilon^2\alpha^2(1-\alpha)^2]}{(1-\epsilon\,\alpha^2)(1 - 2\,\epsilon\,\alpha + \epsilon\,\alpha^2)}\,\mathrm{d}\alpha = 1 + \frac{1}{1-\epsilon}
\tag{A.2}
$$

which is straightforward.

It is also instructive to compute the individual terms in the right-hand side of (3.11) which correspond to the contribution of twist two fields to the OPE. Using Euler's integral representation for the hypergeometric function we find

$$
1 + \frac{1}{1-\epsilon} = 2\sum_{l=0}^{\infty} \binom{4l}{2l}^{-1} \epsilon^{2l}\, F(2l+1, 2l+1; 4l+2; \epsilon).
\tag{A.3}
$$

Each $F(2l+1, 2l+1; 4l+2; \epsilon)$ is, in fact, an elementary function. In particular, the first two terms which provide the contribution of the original field $\phi$ and of the stress–energy tensor $T_2$ to the OPE can be written in the form

$$
\begin{aligned}
&2\int_0^1 \frac{\langle 0|V(x_1, x_2)\phi(x_4 + \alpha\, x_{34})|0\rangle}{c(13)(24)}\,\mathrm{d}\alpha\,\bigg|_{\rho_{34}=0} = 2\,F(1, 1; 2; \epsilon)\\
&\qquad = \frac{2}{\epsilon}\,\ln\frac{1}{1-\epsilon} = 2 + \epsilon + \sum_{n=2}^{\infty}\frac{2\,\epsilon^n}{n+1},\\
&2\,C_1\int_0^1 \frac{\langle 0|V(x_1, x_2)T_2(x_4 + \alpha\, x_{34}, x_{34})|0\rangle}{c(13)(24)}\,\mathrm{d}\alpha\,\bigg|_{\rho_{34}=0} = \frac{\epsilon^2}{3}\,F(3, 3; 6; \epsilon)\\
&\qquad = \frac{60}{\epsilon^2}\left[\left(\frac{1}{\epsilon} - 1 + \frac{\epsilon}{6}\right)\ln\frac{1}{1-\epsilon} - 1 + \frac{\epsilon}{2}\right]\\
&\qquad = \epsilon^2\left\{\frac{1}{3} + \frac{\epsilon}{2} + \sum_{n=2}^{\infty}\frac{(4)_{n-1}(5)_{n-2}\,\epsilon^n}{(3)_{n-2}(7)_{n-1}}\right\}.
\end{aligned}
\tag{A.4}
$$

Proceeding to the general case ($\rho_{34} \neq 0$) we shall use the following generalization of (A.3) (see [10]). Exchange the conformal cross ratios (1.5) (3.9) with the variables $\eta$ and $\overline{\eta}$ related to $\eta_1$ and $\epsilon$ by

$$
\eta\,\overline{\eta} = \eta_1, \qquad \eta + \overline{\eta} = \epsilon + \eta_1, \qquad ((1-\eta)(1-\overline{\eta}) = \eta_2).
\tag{A.5}
$$

(We note that for spacelike $x_{ij}$ the variables $\eta$ and $\overline{\eta}$ are complex conjugate to each other.) In terms of these variables we can write (see equation (3.10) of [10])

$$\frac{2}{X_y^2}\,(4l+1)\int_0^1 \mathrm{d}\alpha\ \alpha^{2l}(1-\alpha)^{2l}\sum_{n=0}^{\infty}\frac{\left(-[\alpha\,(1-\alpha)/4]\,\rho_{34}\,\Box_4\right)^n}{n!\,(2l+1)_n}\,\rho_{34}^l\,(X_y^2)^{l+1}\,C_{2l}^1(\hat{X}_y\cdot\hat{x}_{34})$$
$$= 2\binom{4l}{2l}^{-1}\frac{\eta^{2l+1}\,F(2l+1,2l+1;4l+2;\eta)-\overline{\eta}^{2l+1}\,F(2l+1,2l+1;4l+2;\overline{\eta})}{\eta-\overline{\eta}}. \tag{A.6}$$

We can sum up these expressions applying (A.3); as a result the $\eta_1$-dependent terms present for each $l$ cancel and we end up with

$$\frac{2}{\eta-\overline{\eta}}\times\sum_{l=0}^{\infty}\binom{4l}{2l}^{-1}\{\eta^{2l+1}\,F(2l+1,2l+1;4l+2;\eta)-\overline{\eta}^{2l+1}\,F(2l+1,2l+1;4l+2;\overline{\eta})\}$$
$$=\frac{1}{\eta-\overline{\eta}}\left(\eta+\frac{\eta}{1-\eta}-\overline{\eta}-\frac{\overline{\eta}}{1-\overline{\eta}}\right)=1+\frac{1}{(1-\eta)(1-\overline{\eta})}=1+\frac{1}{1-\epsilon}$$
$$. \tag{A.7}$$

This completes the proof of proposition 3.1. □

## Appendix B. Traces of products of $r^{\mu}_{\nu}(x)$

We shall compute the trace of the product of tensor structures that appears in the numerator of the box diagram with propagator (6.12):

$$B = f^{\lambda_1\mu_1}_{\lambda_2\mu_2}(x_{12})\,f^{\lambda_2\mu_2}_{\lambda_3\mu_3}(x_{23})\,f^{\lambda_3\mu_3}_{\lambda_4\mu_4}(x_{34})\,f^{\lambda_4\mu_4}_{\lambda_1\mu_1}(x_{14}),$$
$$f^{\lambda\mu}_{\lambda'\mu'}(x)=r^{\lambda}_{\lambda'}(x)\,r^{\mu}_{\mu'}(x)-r^{\lambda}_{\mu'}(x)\,r^{\mu}_{\lambda'}(x), \tag{B.1}$$

establishing on the way some useful properties of products of $r^{\mu}_{\nu}(x)=\delta^{\mu}_{\nu}-2\,x^{\mu}\,x_{\nu}/x^2+i\,0\,x^0$ (3.4) (of different arguments) which appear in correlation functions of tensor fields.

We shall use repeatedly the triple-product formula of [24]:

$$r(x_{12})\,r(x_{23})\,r(x_{13})=r(X_{23}),\qquad \text{i.e.}$$
$$r^{\lambda}_{\sigma}(x_{12})\,r^{\sigma}_{\tau}(x_{23})\,r^{\tau}_{\mu}(x_{13})=r^{\lambda}_{\mu}(X_{23}),\qquad X_{23}=\frac{x_{13}}{\rho_{13}}-\frac{x_{12}}{\rho_{12}}. \tag{B.2}$$

Using the identity $r(x)^2=\mathbf{1}$ we find

$$\begin{aligned} R(x_{12},x_{23},x_{34},x_{14}) &:= r(x_{12})\,r(x_{23})\,r(x_{34})\,r(x_{14})\\ &= [r(x_{12})\,r(x_{23})\,r(x_{13})][r(x_{13})\,r(x_{34})\,r(x_{14})]\\ &= r(X_{23})\,r(X_{34}), \end{aligned} \tag{B.3}$$

where $X_{34}\equiv X^1_{34}=x_{14}/\rho_{14}-x_{13}/\rho_{13}$ (cf (3.4)). Using further the relation

$$\mathrm{tr}(r(x)\,r(y))\left(=4\,\frac{(x\cdot y)^2}{x^2\,y^2}+D-4\right)=4\,\frac{(x\cdot y)^2}{x^2\,y^2}\qquad\text{for } D=4 \tag{B.4}$$

we deduce (for $\eta_i$ given by (1.5))

$$\mathrm{tr}(R(x_{12},x_{23},x_{34},x_{14}))=\frac{(2\,X_{23}\cdot X_{34})^2}{X_{23}^2\,X_{34}^2}=\frac{(1-\eta_1-\eta_2)^2}{\eta_1\,\eta_2}. \tag{B.5}$$

A simple algebra allows us to reduce $B$ (B.1) to the difference

$$B=8\,\{[\mathrm{tr}\,R(x_{12},x_{23},x_{34},x_{14})]^2-\mathrm{tr}([R(x_{12},x_{23},x_{34},x_{14})]^2)\}. \tag{B.6}$$

The second term is computed using (B.3) once more:

$$\begin{aligned}
\operatorname{tr}([R(x_{12}, x_{23}, x_{34}, x_{14})]^2) &= \operatorname{tr}(r(X_{23})\, r(X_{34})\, r(X_{23})\, r(X_{34})) \\
&= \operatorname{tr}\left\{ r\left\{ \frac{\rho_{12}\, x_{14} - \rho_{14}\, x_{12}}{\rho_{24}^2} + \frac{\rho_{12}\, x_{13} - \rho_{13}\, x_{12}}{\rho_{23}^2} \right) \right. \\
&\quad \left. \times\, r\left( \frac{\rho_{13}\, x_{14} - \rho_{14}\, x_{13}}{\rho_{34}^2} + \frac{\rho_{12}\, x_{14} - \rho_{14}\, x_{12}}{\rho_{24}^2} \right) \right\} \\
&= \frac{(1 - 2\,\eta_1 - 2\,\eta_2 + \eta_1^2 + \eta_2^2)^2}{\eta_1^2\, \eta_2^2}.
\end{aligned} \tag{B.7}$$

Finally inserting (B.5) and (B.7) we find

$$B = \frac{32}{\eta_1\, \eta_2}\, (1 - 2\,\eta_1 - 2\,\eta_2 + \eta_1^2 + \eta_2^2 + \eta_1\, \eta_2). \tag{B.8}$$

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